THE DISCRIMINANT MATRIX OF A SEMI-SIMPLE ALGEBRA*

BY

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1. **Introduction.** Let $\mathfrak{A}$ be a linear associative algebra with basis e_1, e_2, $\cdots$, e_n over a field $\mathfrak{F}$ in which n has a reciprocal, and let the constants of multiplication be denoted by c_{ijk}. Let $t_1(x)$ and $t_2(x)$ denote, respectively, the first and second traces of x. In a recent paper† the writer called the symmetric matrices

$$T_1 = \| t_1(e_re_s) \| = \| \sum_{i,j} c_{rij}c_{sji} \|, \quad T_2 = \| t_2(e_re_s) \| = \| \sum_{i,j} c_{irj}c_{jsi} \|$$

the *first* and *second discriminant matrices* of $\mathfrak{A}$ relative to the given basis. The relation of these matrices to the discriminant of an algebraic field was shown, so that the names were justified. It was also shown that under a linear transformation of basis,

$$e_i' = \sum a_{ij}e_j, \quad A = (a_{rs}), \quad | A | = a \neq 0,$$

the matrices are transformed cogrediently, i.e.,

$$T_1' = AT_1\bar{A}, \quad T_2' = AT_2\bar{A} \tag{1}$$

where $\bar{A}$ denotes the transpose of A. Thus the ranks of T_1 and T_2 are invariant, and if $\mathfrak{F}$ is a real field, so are the signatures. Other elementary properties of these matrices were discussed and their occurrence in the literature noted.

In the first part of the present paper the behavior of T_1 under transformation of basis is used to establish the existence for every algebra with a principal unit of a normal basis of simple form. This normal basis has a cyclic property generalizing that of the familiar basis 1, i, j, k for quaternions. By means of this normal basis several new theorems in the theory of semi-simple algebras are obtained, e.g., the fact that T_1 and T_2 are identical, and that the first and second characteristic functions are identical.

In the second part of the paper (§4 et seq.) the discriminant matrices of a direct sum, direct product and complete matric algebra are investigated.

2. **The normal basis.** Let us now assume that e_1 is a principal unit so that

* Presented to the Society, April 18, 1930; received by the editors in November, 1930.

† Annals of Mathematics, (2), vol. 32, p. 60.

$$c_{i1j} = c_{1ij} = \delta_{ij}.$$

We denote by τ_{rs} the elements of T_1. By means of the associativity conditions we may write τ_{rs} in the alternative forms*

$$\tau_{rs} = \sum_{i,k} c_{rik}c_{ski} = \sum_{h,k} c_{srh}c_{hkk}.$$

Then if we set $\sum_k c_{hkk} = d_h$, we may write $\tau_{rs} = \sum_h c_{srh}d_h$. In particular

$$\tau_{r1} = \sum_{i,k} c_{rik}c_{1ki} = \sum_{i,k} c_{rik}\delta_{ki} = \sum_i c_{rii} = d_r,$$

$$\tau_{11} = \sum_i c_{1ii} = n.$$

Thus T_1 is of the form

$$\left\|\begin{matrix} n & d_2 & d_3 & \cdots & d_n \\ d_2 & \tau_{22} & \tau_{23} & \cdots & \tau_{2n} \\ d_3 & \tau_{32} & \tau_{33} & \cdots & \tau_{3n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ d_n & \tau_{n2} & \tau_{n3} & \cdots & \tau_{nn} \end{matrix}\right\|.$$

By means of a transformation of matrix

$$A = \left\|\begin{matrix} 1 & 0 & 0 & \cdots & 0 \\ -d_2/n & 1 & 0 & \cdots & 0 \\ -d_3/n & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ -d_n/n & 0 & 0 & \cdots & 1 \end{matrix}\right\|$$

we find that

$$T_1' = AT_1\bar{A} = \left\|\begin{matrix} n & 0 & 0 & \cdots & 0 \\ 0 & \tau_{22}' & \tau_{23}' & \cdots & \tau_{2n}' \\ 0 & \tau_{32}' & \tau_{33}' & \cdots & \tau_{3n}' \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \tau_{n2}' & \tau_{n3}' & \cdots & \tau_{nn}' \end{matrix}\right\|.$$

By the above transformation $e_1' = e_1$ is the principal unit. It is now obvious that the symmetric matrix T_1' can be reduced to a diagonal matrix by transformations in $\mathfrak{F}$ which leave the principal unit e_1 invariant. We have

* MacDuffee, loc. cit., p. 62 (2).

THEOREM 1. *If $\mathfrak{A}$ has a principal unit, a basis can be so chosen that the principal unit is e_1 and*

$$T_1 = \left\| \begin{array}{ccccc} g_1 & 0 & 0 & \cdots & 0 \\ 0 & g_2 & 0 & \cdots & 0 \\ 0 & 0 & g_3 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & g_n \end{array} \right\|, \tag{2}$$

where the g's are in $\mathfrak{F}$ and $g_1 = n$.

Such a basis will be called a *normal basis* for the algebra.

We have seen that when e_1 is a principal unit, $\tau_{r1} = d_r$, so for a normal basis

$$d_1 = n, \; d_2 = d_3 = \cdots = d_n = 0.$$

Hence* $\tau_{rs} = nc_{sr1}$ so that

$$c_{sr1} = \frac{g_r}{n}\delta_{rs}.$$

In the associativity conditions

$$\sum_h c_{rih}c_{phs} = \sum_h c_{prh}c_{his}$$

set $s = 1$ and use the above relation. We obtain

$$\frac{1}{n}\sum_h c_{rih}\delta_{ph}g_p = \frac{1}{n}\sum_h c_{prh}\delta_{hi}g_i,$$

$$g_p c_{rip} = g_i c_{pri} \qquad (i, p, r = 1, 2, \cdots, n). \tag{3}$$

THEOREM 2. *When a normal basis is taken for an algebra $\mathfrak{A}$ with a principal unit, the constants of multiplication are in the relation* (3), *the g's being given by* (2).

It is interesting to note that for quaternions with the familiar basis 1, i, j, k where $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$, the basis is in normal form with $g_1 = 4$, $g_2 = g_3 = g_4 = -4$. The cyclic property of quaternion multiplication is generalized in the cyclic advance of the subscripts in (3).

3. **Some applications.** We shall now suppose that $\mathfrak{A}$ is semi-simple with a normal basis $e_1, e_2, \cdots, e_n$. Then each g_i in (2) is different from zero.

* Cf. these Transactions, vol. 31 (1929), p. 81, Lemma 7.

THEOREM 3. *In every semi-simple algebra, for every basis, the discriminant matrices T_1 and T_2 are identical.*

By definition

$$T_1 = \left\| \sum_{h,k} c_{rkh} c_{shk} \right\|.$$

By means of (3),

$$T_1 = \left\| \sum_{h,k} \frac{g_k}{g_h} c_{hrk} \frac{g_h}{g_k} c_{ksh} \right\|$$

$$= \left\| \sum_{h,k} c_{hrk} c_{ksh} \right\| = T_2.$$

Thus for a normal basis T_1 and T_2 are identical. Since they are transformed cogrediently (1), they are identical for all bases.

Henceforth we shall speak of *the* discriminant matrix of a semi-simple algebra, and denote it by T.

THEOREM 4. *For every semi-simple algebra and for all bases,*

$$S(x) = TR(x)T^{-1}, \tag{4}$$

where $R(x)$ and $S(x)$ are, respectively, the first and second matrices of x, and T is the discriminant matrix.

If $\mathfrak{A}$ has a normal basis, (3) gives

$$c_{ris} g_s = g_i c_{sri} = g_r c_{isr}.$$

Hence

$$S(e_i)T = TR(e_i).$$

Multiplying by x_i and summing for i gives

$$S(x)T = TR(x)$$

where

$$x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$$

is the general number of the algebra. Thus (4) is established for a normal basis.

Under transformation of basis

$$S(x) = A^{-1} S'(x') A, \quad R(x) = \bar{A} R'(x') \bar{A}^{-1},*$$

* Dickson, *Algebren und ihre Zahlentheorie*, p. 38. The $S(x)$ is the transpose of Dickson's S_x.

and by (1), $T=A^{-1}T'\bar{A}^{-1}$. Hence from (4)

$$A^{-1}S'(x')A = A^{-1}T'\bar{A}^{-1}\bar{A}R'(x')\bar{A}^{-1}\bar{A}T'^{-1}A$$
$$= A^{-1}T'R'(x')T'^{-1}A,$$

so that

$$S'(x') = T'R'(x')T'^{-1}.$$

Thus (4) holds for every basis.

THEOREM 5. *For every semi-simple algebra, and for all bases, the first and second characteristic functions are identical.*

By definition* the first and second characteristic functions of the general number are

$$C_1(\omega) = | R(x) - \omega I |,$$
$$C_2(\omega) = | S(x) - \omega I |.$$

The theorem follows immediately from (4).

4. The discriminant matrix of a direct sum. An algebra $\mathfrak{A}$ is the direct sum $\mathfrak{B}\oplus\mathfrak{C}$ of two algebras $\mathfrak{B}$ and $\mathfrak{C}$ if $\mathfrak{A}=\mathfrak{B}+\mathfrak{C}$, $\mathfrak{B}\mathfrak{C}=\mathfrak{C}\mathfrak{B}=0$, $\mathfrak{B}\wedge\mathfrak{C}=0$. It is known that every semi-simple algebra is the direct sum of simple algebras, and the components are unique except for order.

The direct sum of two matrices M_1 and M_2 is the matrix

$$\begin{Vmatrix} M_1 & O \\ O & M_2 \end{Vmatrix}$$

where the O's stand for rectangular blocks of 0's. Let $T(\mathfrak{A})$ denote the discriminant matrix of $\mathfrak{A}$.

THEOREM 6. *If $\mathfrak{A}=\mathfrak{B}\oplus\mathfrak{C}$, a basis for $\mathfrak{A}$ may be so chosen that*

$$T(\mathfrak{A}) = T(\mathfrak{B}) \oplus T(\mathfrak{C}).$$

We choose the basis numbers $e_1, e_2, \cdots, e_n$ of $\mathfrak{A}$ so that $e_1, e_2, \cdots, e_h$ form a basis for $\mathfrak{B}$ and $e_{h+1}, \cdots, e_n$ a basis for $\mathfrak{C}$. Then $e_re_s=0$ unless $r\leqq h$, $s\leqq h$ or $r>h$, $s>h$. Since $T(\mathfrak{A})=\|t(e_re_s)\|$, obviously

$$T(\mathfrak{A}) = \begin{Vmatrix} T(\mathfrak{B}) & 0 \\ 0 & T(\mathfrak{C}) \end{Vmatrix}.$$

COROLLARY 6. *If $\mathfrak{F}$ is a real field, and $\mathfrak{A}=\mathfrak{B}\oplus\mathfrak{C}$, then the signature of $T(\mathfrak{A})$ is the sum of the signatures of $T(\mathfrak{B})$ and $T(\mathfrak{C})$.*

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* Dickson, ibid., p. 37.

5. The discriminant matrix of a direct product. An algebra $\mathfrak{A}$ is the direct product $\mathfrak{B}\times\mathfrak{C}$ of two algebras $\mathfrak{B}$ and $\mathfrak{C}$ if $\mathfrak{A}=\mathfrak{B}\mathfrak{C}$, the order of $\mathfrak{A}$ is the product of the orders of $\mathfrak{B}$ and $\mathfrak{C}$, and if every number of $\mathfrak{B}$ is commutative with every number of $\mathfrak{C}$.

If $P=(p_{rs})$ and $Q=(q_{rs})$ are two square matrices, the direct product* $P\times Q$ is defined to be the matrix

$$\left\| \begin{matrix} Pq_{11} & Pq_{12} \cdots Pq_{1n} \\ Pq_{21} & Pq_{22} \cdots Pq_{2n} \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ Pq_{n1} & Pq_{n2} \cdots Pq_{nn} \end{matrix} \right\|$$

where Pq_{ij} stands for the block of elements

$$\begin{matrix} p_{11}q_{ij} & p_{12}q_{ij} \cdots p_{1n}q_{ij} \\ p_{21}q_{ij} & p_{22}q_{ij} \cdots p_{2n}q_{ij} \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ p_{n1}q_{ij} & p_{n2}q_{ij} \cdots p_{nn}q_{ij}. \end{matrix}$$

THEOREM 7. *If $\mathfrak{A}=\mathfrak{B}\times\mathfrak{C}$, a basis for $\mathfrak{A}$ can be so chosen that*

$$T(\mathfrak{A}) = T(\mathfrak{B}) \times T(\mathfrak{C}).$$

Suppose that $\mathfrak{B}$ has the basis $e_1, e_2, \cdots, e_g$ and the constants of multiplication b_{ijk}, while $\mathfrak{C}$ has the basis $f_1, f_2, \cdots, f_h$ and the constants of multiplication c_{ijk}. Then the numbers $e_{i_1}f_{i_2}$ form a basis for $\mathfrak{A}$, and

$$\begin{aligned} e_{i_1}f_{i_2}e_{j_1}f_{j_2} &= e_{i_1}e_{j_1}f_{i_2}f_{j_2} \\ &= \sum_{\substack{k_1=1,\cdots,g \\ k_2=1,\cdots,h}} b_{i_1j_1k_1}c_{i_2j_2k_2}e_{k_1}f_{k_2}. \end{aligned} \tag{5}$$

Let us denote the basis numbers $e_{i_1}f_{i_2}$ of $\mathfrak{A}$ by the symbols E_i and order them so that

$$i - 1 = g(i_2 - 1) + i_1 - 1, \quad 0 \leqq i_1 - 1 < g. \tag{6}$$

Evidently i_1 and i_2 determine i uniquely and conversely. Then (5) may be written

* A. Hurwitz, Mathematische Annalen, vol. 45, p. 381.
C. Stephanos, Journal de Mathématiques, (5), vol. 6, p. 73.

$$E_iE_j = \sum_{k=1}^{n} D_{ijk}E_k$$

where $D_{ijk}=b_{i_1j_1k_1}c_{i_2j_2k_2}$, the j_1, j_2, k_1, k_2 being determined from j and k by relations similar to (6).

It now follows that

$$\begin{aligned} T(\mathfrak{A}) &= \left\| \sum_{p,q=1}^{gh} D_{rpq}D_{sqp} \right\| \\ &= \left\| \sum_{p_1,q_1,p_2,q_2} b_{r_1p_1q_1}c_{r_2p_2q_2}b_{s_1q_1p_1}c_{s_2q_2p_2} \right\| \\ &= \left\| \sum_{p_1,q_1} b_{r_1p_1q_1}b_{s_1q_1p_1} \sum_{p_2,q_2} c_{r_2p_2q_2}c_{s_2q_2p_2} \right\| \\ &= T(\mathfrak{B}) \times T(\mathfrak{C}). \end{aligned}$$

Corollary 7. *If $\mathfrak{F}$ is a real field, and $\mathfrak{A}=\mathfrak{B}\times\mathfrak{C}$, the signature of $T(\mathfrak{A})$ is equal to the product of the signatures of $T(\mathfrak{B})$ and $T(\mathfrak{C})$.*

As in Theorem 1 bases for $\mathfrak{B}$ and $\mathfrak{C}$ may be so chosen that $T(\mathfrak{B})$ and $T(\mathfrak{C})$ are diagonal matrices. It is evident that $T(\mathfrak{A})$ is now a diagonal matrix whose diagonal elements are the products of the diagonal elements of $T(\mathfrak{B})$ by the diagonal elements of $T(\mathfrak{C})$. Let p_1, p_2 and p denote, respectively, the number of positive terms in the main diagonals of $T(\mathfrak{B})$, $T(\mathfrak{C})$and $T(\mathfrak{A})$. Similarly let n_1, n_2 and n denote the number of negative terms. Then evidently

$$p = p_1p_2 + n_1n_2, \quad n = p_1n_2 + p_2n_1,$$

$$p - n = (p_1 - n_1)(p_2 - n_2).$$

6. **The discriminant matrix of a complete matric algebra.** If $\mathfrak{A}$ is a complete matric algebra of order n^2, basal numbers e_{ij} can be so chosen that

$$e_{ij}e_{kl} = \delta_{jk}e_{il}$$

where δ_{jk} is Kronecker's delta. We shall arrange the e_{ij} in the order e_{11}, e_{22}, $\cdots$, e_{nn} followed by e_{ij}, e_{ji}, $j>i$; $i=1, 2, \cdots; j=i+1, \cdots$. If we denote $T(\mathfrak{A})$ by (τ_{rs}), then

$$\begin{aligned} \tau_{rs} &= t(e_{r_1r_2}e_{s_1s_2}) = t(\delta_{r_2s_1}e_{r_1s_2}) = n\delta_{r_2s_1}t(e_{r_1s_2}) \\ &= n\delta_{r_2s_1}\delta_{r_1s_2}. \end{aligned}$$

Thus $T(\mathfrak{A})$ is in the form

$$\left\|\begin{array}{cccccccc} n & 0\cdots 0 & & & & & & \\ 0 & n\cdots 0 & & & & & & \\ \cdot & \cdot\ \cdot\ \cdot\ \cdot & & & & & & \\ 0 & 0\cdots n & & & & & & \\ & & 0 & n & & & & \\ & & n & 0 & & & & \\ \cdot & \cdot\ \cdot\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & & & 0 & n \\ & & & & & & n & 0 \end{array}\right\| . \tag{7}$$

THEOREM 8. *If $\mathfrak{A}$ is a complete matric algebra, basal numbers may be so chosen that $T(\mathfrak{A})$ is of the form* (7).

COROLLARY 8. *If $\mathfrak{F}$ is a real field and $\mathfrak{A}$ is a complete matric algebra of order n^2, the signature of $T(\mathfrak{A})$ is n.*

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